Quadratic Functions and Their Properties

Quadratic Function: A function that can be written in the form $f\left(x\right)=ax^2+bx+c$, where a, b, and c are real numbers and where $a\neq 0$. The domain of a quadratic function is the set of all real numbers.

Graphing a Quadratic Function Using Transformations

1. Begin with the parent function $f\left(x\right)=x^2$.
2. Rewrite the function in vertex form $f\left(x\right)=a(x-h)^2+k$ by completing the square.
3. Transform with the following:

   a: If a is positive, the graph opens __________. The y-coordinate of the vertex is a __________ value.
   If a is negative, the graph opens __________. The y-coordinate of the vertex is a __________ value.
   If $|a|>1$, the graph is __________ than the graph of $f\left(x\right)=x^2$.
   If $|a|<1$, the graph is __________ than the graph of $f\left(x\right)=x^2$.

   h: h controls the horizontal shift (left and right).

   k: k controls the vertical shift (up and down).

Vertex: $\left(h,k\right)$ Axis of Symmetry: $x=h$

Completing the Square: Figuring out what constant to add to a binomial of the form x^2+bx to make it into a perfect square trinomial, then writing the result in factored form.

Completing the Square for the Binomial x^2+bx

1. Divide the coefficient of the x-term by 2. $\left(\text{Find } \dfrac{b}{2}\right)$.
2. Square the answer from step 1. $\left(\text{Find } \left(\dfrac{b}{2}\right)^2\right)$.
3. Add the result of step 2 to the binomial.
4. Rewrite as a perfect square: $\left(x+\dfrac{b}{2}\right)^2$.

Example: Add the proper constant to the binomial to make it into a perfect square trinomial. Then factor the trinomial.

$x^2+12x+\underline{\qquad}$	$x^2-7x+\underline{\qquad}$

Examples: Add the proper constant to each binomial to make it into a perfect square trinomial. Then factor the trinomial.

a) $x^2+16x+\underline{\qquad}=\left(x\underline{\qquad\quad}\right)^2$ d) $x^2-3x+\underline{\qquad}=\left(x\underline{\qquad\quad}\right)^2$ e) $x^2+\dfrac{4}{3}x+\underline{\qquad}=\left(x\underline{\qquad\quad}\right)^2$

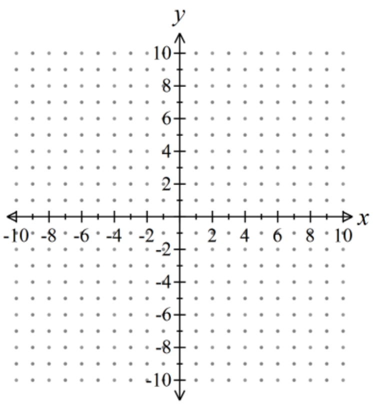
Writing $f(x) = ax^2 + bx + c$ in Vertex Form

1. Group ax^2 and bx together in parentheses.
2. If $a \neq 1$, factor out a from $ax^2 + bx$. Include a negative if the quadratic term is negative.
3. Complete the square (divide b by 2 and square the result). Add the answer inside the parentheses. Keep the equation balanced by adding or subtracting outside the parentheses. (You are adding 0 to one side of the equation.)
4. Write the expression inside the parentheses as a perfect square.

Examples: Write each equation in vertex form. Then find the vertex.

a) $f(x) = x^2 - 8x - 5$ b) $f(x) = 3x^2 + 6x + 1$ c) $y = -x^2 + 4x - 3$

Example: Rewrite the function $f(x) = -2x^2 + 12x - 13$ in vertex form by completing the square. Find the vertex and axis of symmetry, then draw the graph. Find the maximum or minimum value.



The Vertex Formula

By completing the square, we can rewrite $f(x) = ax^2 + bx + c$ as $f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$.

This gives us a quick way to find the vertex when the equation is in standard form:

- The x-coordinate of the vertex is $-\frac{b}{2a}$.
- To find the y-coordinate, plug the x-coordinate into the original equation.

Properties of the Graph of a Quadratic Function

For the function, $f(x) = ax^2 + bx + c,\ a \neq 0$:

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ **Axis of Symmetry:** The line $x = -\frac{b}{2a}$

Parabola opens up if $a > 0$; the vertex is a minimum point.
Parabola opens down if $a < 0$; the vertex is a maximum point.

Examples: Use the vertex formula to locate the vertex and axis of symmetry of the parabola. Determine whether the quadratic function has a maximum or minimum value, then find the value.

a) $f(x) = x^2 - 6x - 8$

b) $f(x) = -4x^2 + 2x + 1$

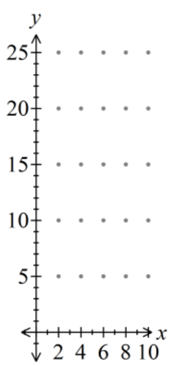
Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

The *y*-intercept is the value of f at $x = 0$; that is, the *y*-intercept is $f(0) = c$.

The *x*-intercepts, if any, are found by solving the quadratic equation $ax^2 + bx + c = 0$.

Discriminant	**Graph of** $\boldsymbol{f(x) = ax^2 + bx + c}$	**Solutions of** $\boldsymbol{ax^2 + bx + c = 0}$
$b^2 - 4ac > 0$	Two *x*-intercepts	Two real solutions
$b^2 - 4ac = 0$	One *x*-intercept	One real solution
$b^2 - 4ac < 0$	No *x*-intercepts	No real solutions

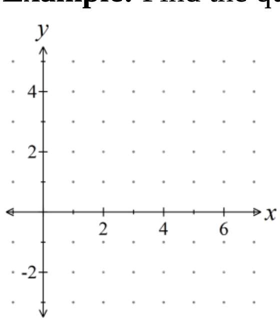
Example: Graph the function $f(x) = x^2 - 10x + 25$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, *y*-intercept, and *x*-intercepts, if any. Determine the domain and range of f and determine where f is increasing and where it is decreasing.



Writing a Quadratic Equation when You Know the Vertex and Another Point

1. Use vertex form: $y = a(x-h)^2 + k$
2. Plug in the vertex for *h* and *k*.
3. Plug in the other point for *x* and *y* (or $f(x)$).
4. Simplify and solve for *a*. (Don't forget order of operations.)
5. Write your final answer by plugging *a*, *h*, and *k* back into vertex form.

Example: Find the quadratic function whose vertex is $(3, -2)$ and whose *y*-intercept is 4. Graph the function.



	Standard Form	**Vertex Form**	**Factored Form**
Equation	$y = ax^2 + bx + c$	$y = a(x-h)^2 + k$	$y = a(x-p)(x-q)$
Vertex	Complete the square and write in vertex form. -or- $x = \frac{-b}{2a}$ Plug the x-coordinate into the equation to get the y-coordinate.	(h,k)	Find average of p and q. $x = \frac{p+q}{2}$ (The x-coordinate of the vertex is at the midpoint of the x-intercepts.) Plug the x-coordinate into the equation to get the y-coordinate.
***y*-intercept**	c (Replace x with zero. Solve for y.)	Replace x with zero. Solve for y.	Replace x with zero. Solve for y.
***x*-intercepts (roots, zeros, solutions)**	Replace y with zero. Solve for x by factoring or quadratic formula.	Replace y with zero. Solve for x by isolating the perfect square and using the square root principle. (Don't forget the ±.)	p and q (Replace y with zero. Solve for x using the zero product property.)

For all forms:

Direction of Opening	Up if a is positive Down if a is negative
Vertical Stretch	a
Counting Pattern (Shortcut)	Start at the vertex. Find more points by counting: $\leftrightarrow 1, \uparrow a$ $\leftrightarrow 1, \uparrow 3a$ $\leftrightarrow 1, \uparrow 5a$ $\leftrightarrow 1, \uparrow 7a$, etc. (If a is negative, move down instead of up.)